Model Answer

Question (1)

(a) Find the root of the equation $x^3 + 3x - 10 = 0$ by using Newton method.

Answer

Newton's method

$$x^3 + 3x - 10 = 0$$

$$f(x) = x^3 + 3x - 10$$

$$f'(x) = 3x^2 + 3$$

$$x_{n+1} = x_n - \frac{x_n^3 + 3x_n - 10}{3x_n^2 + 3} = \frac{2x_n^3 + 10}{3x_n^2 + 3}$$

i	x_i	x_{i+1}	$f(x_{i+1})$
0	1	2	4
1	2	1.733333333	0.407703704
2	1.733333333	1.699395733	0.005950068
3	1.699395733	1.698885604	1.32658E-06
4	1.698885604	1.69888549	6.39488E-14
5	1.69888549	1.69888549	0

$$\boxed{R = 1.69888549}$$

(b) Construct the difference table to the following data.

Hence find interpolation polynomial interpolate the function $y = f(x)$ at these points (0, 1), (0.1, 1.32), (0.2, 1.68), (0.3, 2.08), (0.4, 2.52).

<u>Answer</u>

The difference table

x	y	∂f	$\partial^2 f$	$\partial^3 f$	$\partial^4 f$
0	**1**				
		0.32			
			0.02		
0.1	**1.32**			**0.04**	**-0.06**
		0.34			
0.2	**1.68**				
			0.06		
		0.4		-0.02	
0.3	**2.08**				
			0.04		
		0.44			
0.4	**2.52**				

$$P_3(x) = y_0 + \partial f\,(x - x_0) + \partial^2 f\,(x - x_0)(x - x_1)$$
$$+ \partial^3 f\,(x - x_0)(x - x_1)(x - x_2) + \partial f\,(x - x_0)(x - x_1)(x - x_2)(x - x_3)$$
$$= 1 + 0.32(x - 0) + 0.02(x - 0)(x - 0.1) + 0.04(x - 0)(x - 0.1)(x - 0.2)$$
$$- 0.06(x - 0)(x - 0.1)(x - 0.2)(x - 0.3)$$

Question (2)

(a) Approximate the integrals $\int_0^1 \sqrt{1+x}\, dx$ using Sampson's rule.

Estimate the error by computing the exact value.

<u>Answer</u>

$$\int_0^1 \sqrt{1+x}\, dx$$

Solution: (a) by Trapezoidal Rule

i	x_i	$f(x_i)$	$2\,f(x_i)$
0	0	1	1
1	0.1	1.048808848	2.097617696
2	0.2	1.095445115	2.19089023
3	0.3	1.140175425	2.28035085
4	0.4	1.183215957	2.366431913
5	0.5	1.224744871	2.449489743
6	0.6	1.264911064	2.529822128
7	0.7	1.303840481	2.607680962
8	0.8	1.341640786	2.683281573
9	0.9	1.378404875	2.75680975
10	1	1.414213562	1.414213562

I= 1.21882942

The exact value

$$\int_0^1 \sqrt{1+x}\,dx = \frac{2}{3}\Big[(1+x)^{3/2}\Big]_0^1 = \frac{2}{3}\Big[2^{3/2}-1\Big] = 1.21895141649764$$

Error=1.22E-4

(b) Solve the differential equation $y' = x + y,\quad 0 < x < 1,\quad y(0) = 1$ using Euler method considering $h = 0.2$. Estimate the error at $x = 0.4$ by comparing your result with the exact solution of the problem.

Answer

$$y' = (x + y)\,, \qquad 0 < x < 1, \qquad y(0) = 1$$

Answer:

$$y_{n+1} = y_n + hf(x_n, y_n)$$
$$y_{n+1} = y_n + hf(x_n, y_n)$$
$$= y_n + 0.2[x_n + y_n)]$$
$$= y_n + 0.2[0.2n + y_n)]$$

$$y_{n+1} = y_n + (0.2)^2 n + 0.2y_n)$$

$$y_{n+1} = 1.2y_n + (0.2)^2 n$$

$$\boxed{n = 0,1,2,3,4,\ldots}$$

$$y_1 = 1.2y_0 + (0.04)(0) = 1.2(1) + 0 = 1.2$$

$$y_2 = 1.2y_1 + (0.04)(1) = 1.2(1.2) + (0.04)(1) = 1.48$$

$$y_3 = 1.2y_2 + (0.04)(2) = 1.2(1.48) + (0.04)(2) = 1.856$$

$$y_4 = 1.2y_3 + (0.04)(3) = 1.2(1.856\ ) + (0.04)(3) = 2.3472$$

$$y_5 = 1.2y_4 + (0.04)(4) = 1.2(2.3472) + (0.04)(4) = 2.97664$$

Exact solution

$$y' = (x + y)$$
$$y' - y = x$$
$$e^{-x}y' - ye^{-x} = xe^{-x}$$
$$d(ye^{-x}) = xe^{-x}$$

$$ye^{-x} = \int xe^{-x}dx = -xe^{-x} - e^{-x} + c$$

$$ye^{-x} = -xe^{-x} - e^{-x} + c$$

Substitute by the initial condition

$$ye^{-x} = -xe^{-x} - e^{-x} + c$$

$$1 = 0 - 1 + c \Rightarrow c = 2$$

$$\therefore ye^{-x} = xe^{-x} - e^{-x} + 2$$

$$y = -x - 1 + 2e^{x}$$

X=0.4

$$y = -0.4 - 1 + 2e^{-0.4} = 1.583649395$$

Error=**1.48 - 1.583649395** =0.103649395